Math 142 Lecture 9 Notes

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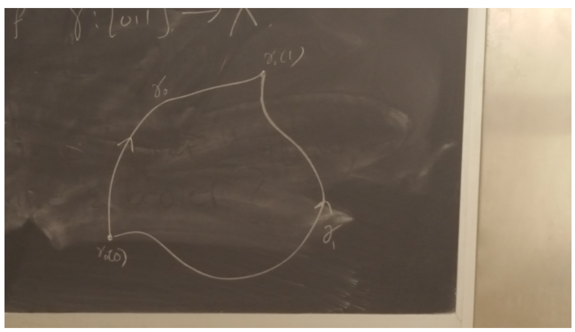
1 Homotopy

1.1 Definition and examples

Recall that a *path* is a continuous function $\gamma : [0, 1] \to X$. The idea here is that we have two paths γ_0 and γ_1 in $\mathbb{R}^2$ with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$; i.e. the paths have the same endpoints.



Our paths are different and may even have different images, but we want to say that one can be continuously deformed into the other.

Intuitively, we want to create a family of paths $\{\gamma_t\}_{t\in[0,1]}$ "from γ_0 to γ_1." You can also say we want to interpolate continuously between γ_0 and γ_1. Think of $\{\gamma_t : t \in [0, 1]\}$ as one function $\gamma : [0, 1] \times [0, 1] \to X$, where $\gamma(s, t) := \gamma_t(s)$.

Definition 1.1. If $f, g : X \to Y$ are continuous, then a *homotopy* F from f to g is a continuous function

$$F : X \times [0,1] \to Y,$$

where

$$F(x,0) = f(x), \qquad F(x,1) = g(x).$$

Here, we way that f is *homotopic* to g and write $f \simeq g$ (or $f \simeq_F g$).

For our paths, we want to fix the start and end, so $\gamma_t(0) = \gamma_t(1) = \gamma_0(1)$ for all $t \in [0,1]$.

Definition 1.2. If $A \subseteq X$ is a subset with $f, g : X \to Y$ continuous such that $f(a) = g(a)$ for all $a \in A$, then a *homotopy* F from f to g *relative to* A is a homotopy F from f to g such that $f(a,t) = f(a)$ for all $a \in A$. We write $f \simeq_F g$ rel A.

Example 1.1. Let f be any continuous function. Then $f \simeq f$ via $F(x,t) = f(x)$ for all $t \in [0,1]$.

Example 1.2. If $S^1 \subseteq \mathbb{C}$ is the set $\{e^{i\theta} : \theta \in \mathbb{R}\}$ and if $\mathrm{id}_{S^1} : S^1 \to S^1$ is the identity, then $\mathrm{id}_{S^1} \simeq_F \mathrm{id}_{S^1}$, where $F(e^{i\theta,t}) = e^{i(\theta+2\pi t)}$. Here, F rotates $e^{i\theta}$ be $2\pi t$ radians, so $F(e^{i\theta},0) = F(e^{i\theta},1) = \mathrm{id}_{S'}$.

1.2 Homotopy on convex sets

Definition 1.3. A set $A \subseteq \mathbb{R}^n$ is *convex* if for all $x, y \in A$, the line segment $\{(1+t)x + ty : t \in [0,1]\} \subseteq A$.

Example 1.3. If $Y \subseteq \mathbb{R}^n$ is convex and $f, g : X \to Y$ are continuous, then $F(x,t) = (1-t)f(x) + tg(x)$ is a homotopy from f to g called the *straight line homotopy.*

Example 1.4. If Y is convex, $p \in Y$, $f : X \to Y$ is continuous, and $g : X \to Y$ is $g(x) = p$ for all $x \in X$, then $f \simeq g$ via the straight line homotopy.

A more general notion than a set A being convex is the notion of a set being *star-shaped*, which means that there is a point $x \in A$ such that the line segment connecting x to any $y \in A$ is contained in A.

Example 1.5. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$, let $f : X \to X$ be the identity, and let $g : X \to X$ be $g(x) = x/\|x\|$. Then $f \simeq_F g$, where $F(x,t) = (1-t)x + t(x/\|x\|)$. Note that X is not convex, but the line segment between x and $x/\|x\|$ is in X.

1.3 Properties of homotopy

Homotopy defines a sort of equivalence between continuous functions.

Proposition 1.1. *The relation $f \simeq g$ on the set of continuous functions form X to Y is an equivalence relation. (Similarly, $f \simeq g$ rel A is also an equivalence relation.)*

Proof. We check the three parts of the definition of an equivalence relation:

1. $f \simeq f$ by our previous example.
2. If $f \simeq_F g$, then $g \simeq_G f$, where $G(x,t) = F(x, 1-t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then let
$$H(x,t) = \begin{cases} F(x,2t) & t \in [0,1/2] \\ G(x,2t-1) & t \in (1/2,1]. \end{cases}$$
Then $f \simeq_H h$. $\square$

Compositions of continuous functions preserve homotopy in the ways you would want.

Proposition 1.2. *If we have $f, g : X \to Y$ with $f \simeq_F g$ and $h : Y \to Z$, then $h \circ f \simeq h \circ g$. If $j : Z \to X$, then $f \circ j \simeq g \circ j$.*

Proof. For the first part, use the homotopy $h \circ F$. For the second part, use the homotopy $G(x,t) = F(j(x),t)$. $\square$

1.4 Homotopy equivalence of spaces

Definition 1.4. If X and Y are topological spaces, then they are *homotopy equivalent* if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The function f is called a *homotopy equivalence* from $X \to Y$, g is its *homotopy inverse*, and we write $X \simeq Y$.

Proposition 1.3. *The relation $X \simeq Y$ is an equivalence relation.*

Proof. We check the three parts of the definition of an equivalence relation:

1. $X \simeq_{\mathrm{id}_X} X$.
2. Symmetry is built into the definition.
3. If $X \simeq_f Y$ with inverse g' and $Y \simeq_g Z$ with inverse g', then
$$\begin{aligned} (f' \circ g') \circ (g \circ f) &= f' \circ (g' \circ g) \circ f \\ &\simeq f' \circ \mathrm{id}_Y \circ f \end{aligned}$$

$$
\begin{aligned}
&= f' \circ f \\
&\simeq \mathrm{id}_X \,.
\end{aligned}
$$

Similarly, $(g \circ f) \circ (f' \circ g) \simeq \mathrm{id}_Z$, so $g \circ f$ is a homotopy equivalence from X to Z with homotopy inverse $f' \circ g'$. □

Definition 1.5. If $A \subseteq X$, let $i : A \to X$ be the inclusion map ($a \mapsto a$). If the map i is a homotpoy equivalence (with homotopy inverse $f : X \to X$), then we call the map $i \circ f : X \to X$ a *deformation retract* (or *deformation retraction*) of X onto A.

Example 1.6. $\mathbb{R}^2 \setminus \{(0,0)\}$ deformation retracts onto the unit circle.

Definition 1.6. If $A = \{x\} \subseteq X$, and there exists a deformation retract of X onto A, then we say that X is *contractible*.

Example 1.7. Convex subsets Y of $\mathbb{R}^n$ are contractible. If $p \in Y$, then $\mathrm{id}_X \simeq i \circ f$, where $i : \{p\} \to Y$ is the inclusion map, and $f : Y \to P$ sends $y \mapsto p$.